



Research/Technical Note

Notes on “Some Properties of L -fuzzy Approximation Spaces on Bounded Integral Residuated Lattices”

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Abstract: In this note, we continue the works in the paper [Some properties of L -fuzzy approximation spaces on bounded integral residuated lattices", Information Sciences, 278, 110-126, 2014]. For a complete involutive residuated lattice, we show that the L -fuzzy topologies generated by a reflexive and transitive L -relation satisfy $(TC)_L$ or $(TC)_R$ axioms and the L -relations induced by two L -fuzzy topologies, which are generated by a reflexive and transitive L -relation, are all the original L -relation; and give out some conditions such that the L -fuzzy topologies generated by two L -relations, which are induced by an L -fuzzy topology, are all the original L -fuzzy topology.

Keywords: Involutive Residuated Lattice, L -relation, L -fuzzy Topology, L -fuzzy Approximation Space

1. Introduction

A residuated lattice (see [1, 10]) is an algebra $L = (L, \wedge, \vee, \cdot, \rightarrow, \leftarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(L1) (L, \wedge, \vee) is a lattice,

(L2) $(L, \cdot, 1)$ is a monoid, i.e., is associative and $x \cdot 1 = 1 \cdot x = x$ for any $x \in L$,

(L3) $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$ if and only if $y \leq x \leftarrow z$ for any $x, y, z \in L$.

A residuated lattice with a constant 0 is called an FL -algebra. If $x \leq 1$ for all $x \in L$, then L is called integral residuated lattice. An FL -algebra L , which satisfies the condition $0 \leq x \leq 1$ for all $x \in L$, is called an FL_w -algebra or a bounded integral residuated lattice (see [1]).

We adopt the usual convention of representing the monoid operation by juxtaposition, writing ab for $a \cdot b$.

Let L be a bounded integral residuated lattice. Define two negations on L , \neg^L and \neg^R :

$$\neg^L x = x \rightarrow 0, \quad \neg^R x = x \leftarrow 0 \quad \forall x \in L.$$

A bounded residuated lattice L is called an involutive residuated lattice (see [3]) if

$$\neg^L \neg^R x = \neg^R \neg^L x \quad \forall x \in L.$$

In the sequel, unless otherwise stated, L always represents any given complete involutive residuated lattice with maximal element 1 and minimal element 0.

Definition 1.1 (see Liu and Luo [5]). Let $\tau \subseteq L^X$ and J be an index set. If τ satisfies the following three conditions:

(LFT1) $0_X, 1_X \in \tau$,

(LFT2) $\mu, \nu \in \tau \Rightarrow \mu \wedge \nu \in \tau$,

(LFT3) $\mu_j \in \tau (j \in J) \Rightarrow \bigvee_{j \in J} \mu_j \in \tau$,

then τ is called an L -fuzzy topology on X and (L^X, τ) L -fuzzy topological space. Every element in τ is called an open subset in L^X .

When $L = [0, 1]$, an L -fuzzy topological space (L^X, τ) is

also called an F -topological space.

Let $\tau'_L = \{\neg^L \mu \mid \mu \in \tau\}$ and $\tau'_R = \{\neg^R \mu \mid \mu \in \tau\}$. The elements of τ'_L and τ'_R are, respectively, called left closed subsets and right closed subsets in L^X (see Wang et al. [12]).

Definition 1.2 (Wang and Liu [11], Wang et al. [12]). Let τ be an L -fuzzy topology on X and μ L -fuzzy subset of X . The interior, left closure and right closure of μ w.r.t τ are, respectively, defined by

$$\text{int}(\mu) = \bigvee \{\eta \mid \eta \leq \mu, \eta \in \tau\},$$

$$cl_L(\mu) = \bigwedge \{\xi \mid \mu \leq \xi, \xi \in \tau'_L\},$$

$$cl_R(\mu) = \bigwedge \{\zeta \mid \mu \leq \zeta, \zeta \in \tau'_R\}.$$

int , cl_L and cl_R are, respectively, called the interior, left closure and right closure operators.

For the sake of convenience, we denote $\text{int}(\mu)$, $cl_L(\mu)$ and $cl_R(\mu)$ by μ° , μ_L^- and μ_R^- , respectively.

Zhang et al. [14, 15] studied some properties of rough sets and rough approximation operators, Ouyang et al. [6, 7] investigated some generalized models of fuzzy rough sets, Liu and Lin [4] considered the intuitionistic fuzzy rough set model, Wu et al. [13] discussed the axiomatic characterizations of fuzzy rough approximation operators, Radzikowska and Kerre [9] studied L -fuzzy rough sets and lower (upper) L -fuzzy approximation based on commutative residuated lattices. Recently, Wang et al. [12] discussed the notion of left (right) lower and left (right) upper L -fuzzy rough approximation based on complete bounded integral residuated lattices.

Definition 1.3 (Wang et al. [12]). Let R be an L -relation on X . A pair (X, R) is called an L -fuzzy approximation space. Define the following four mappings $R \downarrow_L$, $R \uparrow_L$, $R \downarrow_R$, $R \uparrow_R$: $L^X \rightarrow L^X$, called a left lower, left upper, right lower, and right upper L -fuzzy rough approximation operators, respectively, as follows: for every $\mu \in L^X$ and $x \in X$,

$$R \downarrow_L(\mu)(x) = \bigwedge_{y \in X} (R(x, y) \rightarrow \mu(y)),$$

$$R \uparrow_L(\mu)(x) = \bigvee_{y \in X} \mu(y) R(y, x),$$

$$R \downarrow_R(\mu)(x) = \bigwedge_{y \in X} (R(y, x) \leftarrow \mu(y)),$$

$$R \uparrow_R(\mu)(x) = \bigvee_{y \in X} R(x, y) \mu(y).$$

$R \downarrow_L(\mu)$, $R \uparrow_L(\mu)$, $R \downarrow_R(\mu)$ and $R \uparrow_R(\mu)$ are called left lower, left upper, right lower, and right upper L -fuzzy rough approximations of μ , respectively.

A pair $(\lambda, \xi) \in L^X \times L^X$ such that $\lambda = R \downarrow_L(\mu)$ ($\lambda = R \downarrow_R(\mu)$) and $\xi = R \uparrow_L(\mu)$ ($\xi = R \uparrow_R(\mu)$) for some

$\mu \in L^X$, is called a left (right) L -fuzzy rough set in (X, R) .

When $L = [0, 1]$, L -fuzzy rough approximation operators, L -fuzzy approximation space and left (right) L -fuzzy rough sets are, respectively, called fuzzy rough approximation operators, fuzzy approximation space and left (right) fuzzy rough sets.

2. The L -fuzzy Topologies Generated by a Reflexive and Transitive L -relation

In this section, we supplement some properties of the L -fuzzy topologies generated by a reflexive and transitive L -relation.

If R is a reflexive and transitive L -relation on X , then it follows from Theorem 6.1 in [12] that

$$\tau_1 = \{\xi \mid R \downarrow_L(\xi) = \xi, \xi \in L^X\},$$

$$\tau_2 = \{\xi \mid R \downarrow_R(\xi) = \xi, \xi \in L^X\}$$

are all L -fuzzy topologies on X and $R \downarrow_L$ and $R \downarrow_R$ are just the interior operators w.r.t τ_1 and τ_2 , respectively. Here, τ_1 and τ_2 are called the L -fuzzy topologies generated by the L -relation R or by left lower L -fuzzy rough approximation operator $R \downarrow_L$ and right lower L -fuzzy rough approximation operator $R \downarrow_R$, respectively.

Theorem 2.1. If R is a reflexive and transitive L -relation on X , then

$$\tau_1 = \{\neg^L R \uparrow_R(\xi) \mid \xi \in L^X\},$$

$$\tau_2 = \{\neg^R R \uparrow_L(\xi) \mid \xi \in L^X\},$$

$R \uparrow_R$ and $R \uparrow_L$ are, respectively, the right closure operator w.r.t τ_1 and the left closure operator w.r.t τ_2 .

Proof. When L is an involutive residuated lattice, $\neg^R \neg^L \mu = \neg^L \neg^R \mu = \mu$ for any $\mu \in L^X$.

If R is a reflexive and transitive L -relation on X , then it follows from Theorem 4.1(5) and Remark 5.2 in [12] that

$$\begin{aligned} R \downarrow_L(\neg^L R \uparrow_R(\xi)) &= \neg^L (R \uparrow_R(R \uparrow_R(\xi))) \\ &= \neg^L (R \uparrow_R(\xi)) \quad \forall \xi \in L^X, \end{aligned}$$

i.e., $\neg^L R \uparrow_R(\xi) \in \tau_1$ for any $\xi \in L^X$. If $\xi \in \tau_1$ and $\mu \in L^X$, then it follows from Theorems 3.1(3) and 4.1(5) in [12] that

$$\begin{aligned} \xi &= R \downarrow_L(\xi) = R \downarrow_L(\neg^L \neg^R \xi) \\ &= \neg^L (R \uparrow_R(\neg^R \xi)) = \neg^L R \uparrow_R(\eta), \end{aligned}$$

$$\begin{aligned} \mu_R^- &= \neg^R (\neg^L \mu)^\circ \\ &= \neg^R (R \downarrow_L(\neg^L \mu)) = \neg^R (\neg^L R \uparrow_R(\mu)) \\ &= \neg^R \neg^L (R \uparrow_R(\mu)) = R \uparrow_R(\mu), \end{aligned}$$

where $\eta = \neg^R \xi$. So, $\tau_1 = \{\neg^L R \uparrow_R(\eta) | \eta \in L^X\}$ and $R \uparrow_R$ is the right closure operator w.r.t. τ_1 .

Similarly, we can show that $\tau_2 = \{\neg^R R \uparrow_L(\eta) | \eta \in L^X\}$ and $R \uparrow_L$ is the left closure operator w.r.t. τ_2 .

The theorem is proved.

Recently, Qin et al. [2, 8] studied the topological properties of fuzzy rough sets. The following left and right (TC) axioms are generalizations of (TC) axiom in [8].

(TC)_L axiom: for any $x, y \in X$ and $\mu \in \tau$ there exists $\mu^* \in \tau$ such that $\mu^*(x) = 0$ and

$$\mu^*(y) \rightarrow \mu^*(x) \leq \mu(y) \rightarrow \mu(x).$$

(TC)_R axiom: for any $x, y \in X$ and $\nu \in \tau$ there exists $\nu^* \in \tau$ such that $\nu^*(y) = 0$ and

$$\nu^*(x) \leftarrow \nu^*(y) \leq \nu(x) \leftarrow \nu(y).$$

Theorem 2.2. If R is a reflexive and transitive L -relation on X , then the L -fuzzy topologies τ_1 and τ_2 , generated by R , satisfy (TC)_R and (TC)_L axioms, respectively.

Proof. For any $x, y \in X$ and $\mu \in \tau_1$, let

$$\mu^* = \neg^L (R_{\tau_1}^R \uparrow_R (1_{\{y\}})),$$

then

$$\mu^*(y) = \neg^L (R_{\tau_1}^R \uparrow_R (1_{\{y\}}))(y)$$

$$= \neg^L R_{\tau_1}^R(y, y) = \neg^L 1 = 0,$$

$$\mu^*(x) \leftarrow \mu^*(y) = \neg^L (R_{\tau_1}^R \uparrow_R (1_{\{y\}}))(x) \leftarrow 0$$

$$= \neg^R \neg^L R_{\tau_1}^R(x, y) = R_{\tau_1}^R(x, y)$$

$$= \wedge_{\xi \in \tau_1} (\xi(x) \leftarrow \xi(y)) \leq \mu(x) \leftarrow \mu(y),$$

i.e., τ_1 satisfies (TC)_R axiom; for any $\nu \in \tau_2$, let

$$\nu^* = \neg^R (R_{\tau_2}^L \uparrow_L (1_{\{x\}})),$$

Then

$$\nu^*(x) = \neg^R (R_{\tau_2}^L \uparrow_L (1_{\{x\}}))(x)$$

$$= \neg^R R_{\tau_2}^L(x, x) = \neg^R 1 = 0,$$

$$\nu^*(y) \rightarrow \nu^*(x) = \neg^R (R_{\tau_2}^L \uparrow_L (1_{\{x\}}))(y) \rightarrow 0$$

$$= \neg^L \neg^R R_{\tau_2}^L(x, y) = R_{\tau_2}^L(x, y)$$

$$= \wedge_{\xi \in \tau_2} (\xi(y) \rightarrow \xi(x)) \leq \nu(y) \rightarrow \nu(x),$$

i.e., τ_2 satisfies (TC)_L axiom.

The theorem is proved.

3. The L -relations Induced by an L -fuzzy Topology

In this section, we supplement some properties of the L -relations induced by an L -fuzzy topology.

Let τ be an L -fuzzy topology on X . For any $x, y \in X$, let

$$R_{\tau}^L(x, y) = \wedge_{\mu \in \tau} (\mu(y) \rightarrow \mu(x)),$$

$$R_{\tau}^R(x, y) = \wedge_{\mu \in \tau} (\mu(x) \leftarrow \mu(y)).$$

Clearly, R_{τ}^L and R_{τ}^R are reflexive L -relations on X . Moreover, it follows from Theorem 2.1(5) in [12] that

$$\begin{aligned} R_{\tau}^L(x, y) R_{\tau}^L(y, z) &= \{\wedge_{\mu \in \tau} (\mu(y) \rightarrow \mu(x))\} \{\wedge_{\nu \in \tau} (\nu(z) \rightarrow \nu(y))\} \\ &\leq \wedge_{\mu \in \tau} \{\mu(y) \rightarrow \mu(x)\} \{\mu(z) \rightarrow \mu(y)\} \\ &\leq \wedge_{\mu \in \tau} (\mu(z) \rightarrow \mu(x)) \\ &= R_{\tau}^L(x, z) \quad \forall x, y, z \in X, \end{aligned}$$

$$\begin{aligned} R_{\tau}^R(x, y) R_{\tau}^R(y, z) &= \{\wedge_{\mu \in \tau} (\mu(x) \leftarrow \mu(y))\} \{\wedge_{\nu \in \tau} (\nu(y) \leftarrow \nu(z))\} \\ &\leq \wedge_{\mu \in \tau} \{\mu(x) \leftarrow \mu(y)\} \{\mu(y) \leftarrow \mu(z)\} \\ &\leq \wedge_{\mu \in \tau} (\mu(x) \leftarrow \mu(z)) \\ &= R_{\tau}^R(x, z) \quad \forall x, y, z \in X. \end{aligned}$$

Thus, R_{τ}^L and R_{τ}^R are all transitive L -relations on X . Let

$$R_{\tau}(x, y) = R_{\tau}^R(x, y) \wedge R_{\tau}^L(x, y)$$

$$= \wedge_{\mu \in \tau} \{\mu(x) \leftarrow \mu(y)\} \wedge \{\mu(y) \rightarrow \mu(x)\} \quad \forall x, y \in X.$$

It is easy to see that $R_{\tau} = R_{\tau}^R \wedge R_{\tau}^L$ is also a reflexive and transitive L -relations on X .

Theorem 3.1. If R is a reflexive and transitive L -relation on X , then

$$R = R_{\tau_1}^R = R_{\tau_2}^L.$$

Proof. For any $x, y \in X$, by virtue of Definitions 1.2 and 1.3 and Theorem 2.1, we see that

$$\begin{aligned} R(x, y) &= R \uparrow_R (1_{\{y\}})(x) = (1_{\{y\}})_R^-(x) \\ &= \wedge \{\neg^R \xi(x) | 1_{\{y\}} \leq \neg^R \xi, \xi \in \tau_1\} \\ &= \wedge \{\neg^R \xi(x) | \neg^R \xi(y) = 1, \xi \in \tau_1\} \\ &= \wedge \{\xi(x) \leftarrow 0 | \xi(y) = 0, \xi \in \tau_1\} \\ &= \wedge \{\xi(x) \leftarrow \xi(y) | \xi(y) = 0, \xi \in \tau_1\} \\ &\geq \wedge \{\xi(x) \leftarrow \xi(y) | \xi \in \tau_1\} = R_{\tau_1}^R(x, y); \end{aligned}$$

$$\begin{aligned} R(x, y) &= R \uparrow_L (1_{\{x\}})(y) = (1_{\{x\}})_L^-(y) \\ &= \wedge \{\neg^L \xi(y) | 1_{\{x\}} \leq \neg^L \xi, \xi \in \tau_2\} \\ &= \wedge \{\neg^L \xi(y) | \neg^L \xi(x) = 1, \xi \in \tau_2\} \\ &= \wedge \{\xi(y) \rightarrow 0 | \xi(x) = 0, \xi \in \tau_2\} \\ &= \wedge \{\xi(y) \rightarrow \xi(x) | \xi(x) = 0, \xi \in \tau_2\} \\ &\geq \wedge \{\xi(y) \rightarrow \xi(x) | \xi \in \tau_2\} = R_{\tau_2}^L(x, y). \end{aligned}$$

Thus, $R \geq R_{\tau_1}^R$ and $R \geq R_{\tau_2}^L$.

On the other hand, $R \downarrow_L$ and $R \uparrow_R$ are, respectively, the interior and right closure operators w.r.t. τ_1 and $R \downarrow_R$ and $R \uparrow_L$ are, respectively, the interior and left closure operators w.r.t. τ_2 . Thus, by virtue Theorem 3.1(3) and Remark 5.2 in [12], we can see that

$$\begin{aligned} R \uparrow_R \{\neg^R(R \downarrow_L(\mu))\} &= \{\neg^R(R \downarrow_L(\mu))\}_R^- \\ &= \neg^R\{\neg^L\neg^R(R \downarrow_L(\mu))\}^o = \neg^R\{R \downarrow_L(\mu)\}^o \\ &= \neg^R\{R \downarrow_L(R \downarrow_L(\mu))\} \\ &= \neg^R(R \downarrow_L(\mu)) \forall R \downarrow_L(\mu) \in \tau_1, \\ R \uparrow_L \{\neg^L(R \downarrow_R(\mu))\} &= \{\neg^L(R \downarrow_R(\mu))\}_L^- \\ &= \neg^L\{\neg^R\neg^L(R \downarrow_R(\mu))\}^o = \neg^L\{R \downarrow_R(\mu)\}^o \\ &= \neg^L\{R \downarrow_R(R \downarrow_R(\mu))\} \\ &= \neg^L(R \downarrow_R(\mu)) \forall R \downarrow_R(\mu) \in \tau_2. \end{aligned}$$

So, it follows from the proof of Theorem 7.2 in [12] that $R \leq R_{\tau_1}^R$ and $R \leq R_{\tau_2}^L$.

Therefore, $R = R_{\tau_1}^R = R_{\tau_2}^L$.

The theorem is proved.

This result shows that the reflexive and transitive L -relations $R_{\tau_1}^R$ and $R_{\tau_2}^L$ induced by, respectively, the L -fuzzy topologies τ_1 and τ_2 are all the original reflexive and transitive L -relation.

For any $\mu \in L^X$ and $R \in L^{X \times X}$,

$$\mu = \bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}}).$$

Thus, by Definition 1.3 and Theorem 4.1(3) in [12], we see that

$$\begin{aligned} R \uparrow_L(\mu) &= \bigvee_{x \in X} R \uparrow_L(\mu(x)_X \wedge 1_{\{x\}}) \\ &= \bigvee_{x \in X} \mu(x)_X \cdot R \uparrow_L(1_{\{x\}}), \\ R \uparrow_R(\mu) &= \bigvee_{x \in X} R \uparrow_R(\mu(x)_X \wedge 1_{\{x\}}) \\ &= \bigvee_{x \in X} R \uparrow_L(1_{\{x\}}) \cdot \mu(x)_X. \end{aligned}$$

Theorem 3.2. Let τ be an L -fuzzy topology on X and J index set. Then the following properties hold.

(1) If τ satisfies $(TC)_L$ axiom and the left closure operator w.r.t. τ satisfies the following two conditions:

$$(CL1) \quad (\bigvee_{j \in J} \mu_j)_L^- = \bigvee_{j \in J} (\mu_j)_L^- \quad \forall \mu_j \in L^X,$$

$$(CL2) \quad (a \wedge 1_{\{x\}})_L^- = a \cdot (1_{\{x\}})_L^- \quad \forall a \in L, x \in X,$$

then $R_{\tau}^L \uparrow_L$ and $R_{\tau}^L \downarrow_R$ are, respectively, just the left closure operator and the interior operator w.r.t. τ and

$$\tau = \{\xi \mid R_{\tau}^L \downarrow_R(\xi) = \xi, \xi \in L^X\}.$$

(2) If τ satisfies $(TC)_R$ axiom and the right closure operator w.r.t. τ satisfies the following two conditions:

$$(CR1) \quad (\bigvee_{j \in J} \mu_j)_R^- = \bigvee_{j \in J} (\mu_j)_R^- \quad \forall \mu_j \in L^X,$$

$$(CR2) \quad (a \wedge 1_{\{x\}})_R^- = (1_{\{x\}})_R^- \cdot a \quad \forall a \in L, x \in X,$$

then $R_{\tau}^R \uparrow_R$ and $R_{\tau}^R \downarrow_L$ are, respectively, just the right closure operator and the interior operator w.r.t. τ and

$$\tau = \{\xi \mid R_{\tau}^R \downarrow_L(\xi) = \xi, \xi \in L^X\}$$

Proof. We only prove (1).

If τ satisfies $(TC)_L$ axiom and the left closure operator w.r.t. τ satisfies the conditions (CL1) and (CL2), then it follows from Definition 1.3 and the proof of Theorem 3.1 that

$$\begin{aligned} R_{\tau}^L \uparrow_L(1_{\{x\}})(y) &= R_{\tau}^L(x, y) = \bigwedge_{\mu \in \tau} (\mu(y) \rightarrow \mu(x)) \\ &= \bigwedge \{\mu(y) \rightarrow \mu(x) \mid \mu(x) = 0, \mu \in \tau\} \\ &= (1_{\{x\}})_L^-(y) \quad \forall x, y \in X, \end{aligned}$$

i.e., $(1_{\{x\}})_L^- = R_{\tau}^L \uparrow_L(1_{\{x\}})$ for any $x \in X$. Thus, for any $\mu \in L^X$, we have that

$$\begin{aligned} \mu_L^- &= \{\bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}})\}_L^- \\ &= \bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}})_L^- = \bigvee_{x \in X} \mu(x)_X \cdot (1_{\{x\}})_L^- \\ &= \bigvee_{x \in X} \mu(x)_X \cdot R_{\tau}^L \uparrow_L(1_{\{x\}}) \\ &= \bigvee_{x \in X} R_{\tau}^L \uparrow_L(\mu(x)_X \wedge 1_{\{x\}}) \\ &= R_{\tau}^L \uparrow_L \{\bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}})\} \\ &= R_{\tau}^L \uparrow_L(\mu), \end{aligned}$$

i.e., $R_{\tau}^L \uparrow_L$ is just the left closure operator w.r.t. τ . By Theorems 3.1(2) and 4.1(5) in [12], we see that

$$\begin{aligned} \mu^o &= \neg^R(\neg^L \mu)_L^- = \neg^R R_{\tau}^L \uparrow_L(\neg^L \mu) \\ &= R_{\tau}^L \downarrow_R(\neg^R \neg^L \mu) = R_{\tau}^L \downarrow_R(\mu) \quad \forall \mu \in L^X, \end{aligned}$$

i.e., $R_{\tau}^L \downarrow_R$ is just the interior closure operator w.r.t. τ . Therefore,

$$\tau = \{\xi \mid R_{\tau}^L \downarrow_R(\xi) = \xi, \xi \in L^X\}.$$

The theorem is proved.

This result shows that the L -topologies generated by two reflexive and transitive L -relations R_{τ}^L and R_{τ}^R , which are induced by an L -topology τ , on X are all the original L -topology τ when τ satisfies some conditions.

Moreover, if τ satisfies (CL1) or (CR1), then it follows from Remark 2.1 and Theorem 3.1(2) in [12] that

$$\begin{aligned} (\bigwedge_{j \in J} \mu_j)^o &= \neg^R(\neg^L(\bigwedge_{j \in J} \mu_j))_L^- \\ &= \neg^R(\bigvee_{j \in J} \neg^L \mu_j)_L^- = \neg^R \{\bigvee_{j \in J} (\neg^L \mu_j)_L^-\} \\ &= \bigwedge_{j \in J} \neg^R(\neg^L \mu_j)_L^- = \bigwedge_{j \in J} (\mu_j)^o \quad \forall \mu_j \in L^X, \end{aligned}$$

i.e., the interior operator int of τ distributes over arbitrary intersection of L -fuzzy sets. Thus, the intersection of arbitrarily many open subsets is still an open subset.

4. Conclusions and Future Work

In this note, we continue the works in [12]. For a complete involutive residuated lattice, we have supplemented some properties of the L -fuzzy topologies generated by a reflexive and transitive L -relation; showed that the L -fuzzy topologies generated by a reflexive and transitive L -relation satisfy $(\text{TC})_L$ or $(\text{TC})_R$ axioms; and given out some conditions such that the L -fuzzy topologies generated by two L -relations, which are induced by an L -fuzzy topology, are all the original L -fuzzy topology.

In a forthcoming paper, we will discuss the relationships between the L -fuzzy topological spaces and the L -fuzzy rough approximation spaces on a complete involutive residuated lattice.

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